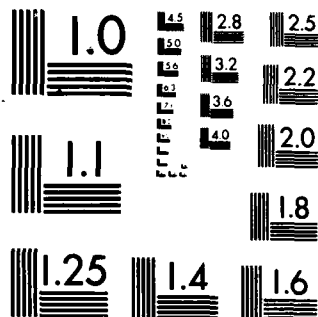


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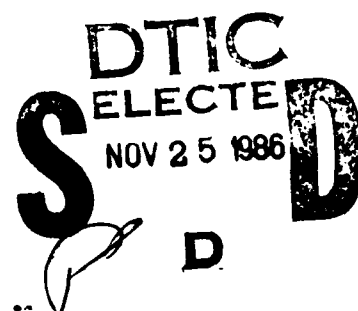
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FRACTIONAL FACTORIAL DESIGNS IN THE FORM  
OF INCOMPLETE ORTHOGONAL ARRAYS

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*Key Words and Phrases:* Fractional factorial; orthogonal arrays;  
orthogonal polynomial model; balanced arrays.

ABSTRACT

In this paper we study certain fractional factorial designs, which are known in the literature as incomplete orthogonal arrays. We indicate situations in which these designs can be of practical interest and study their statistical properties.

1. INTRODUCTION

The problem that we will consider in this paper deals with symmetrical factorial designs based on  $k$  factors each at  $s$  levels. Potentially there are  $s^k$  possible combinations at which observations can be made in such experiments. In many practical situations it is undesirable to make this many observations. This introduces a need to select some of the possible combinations at which one or more observations should be obtained. Such experiments, in which one or more of the possible combinations are not used, are known as fractional factorial experiments. The choice of the combinations at which observations are to be made is



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a difficult problem. There are statistical considerations (e.g., the proposed model, the experimenters objective), but also practical considerations (e.g., financial limitations, time considerations) that can influence this choice. In a series of three papers Rao (1946, 1947a, 1947b) inspired many later researchers by introducing and studying certain fractions with many desirable statistical properties. These fractions are now known as orthogonal arrays. A formal definition will be given in Section 2. For the current status of this subject we refer the reader to Hedayat and Stufken (1986b).

The inspiration for this paper is that there are practical situations for which certain specific combinations have to be excluded from experimentation. For example, it may be desirable that combinations in which 2 or more factors appear at certain levels are excluded. Safety considerations or financial reasons are potential causes for such a restriction. This paper studies certain fractions in which some undesirable combinations do not appear, but in which the remaining structure is such that under the given restriction the fractions are 'as close as possible' to an orthogonal array.

Example 1.1: Suppose we are considering an experiment based on 3 factors, each at 4 levels denoted by 0, 1, 2 and 3. It may be desirable to have a fraction in the form of an orthogonal array of strength 2, except that due to practical considerations it may be undesirable to have any treatment combination in which more than one factor appears at level 2 or 3. A viable design under the given restriction is exhibited below.

```

0 0 0 0 1 1 1 1 2 2 3 3
0 1 2 3 0 1 2 3 0 1 0 1
2 3 1 0 3 2 0 1 1 0 0 1

```

The combinations to be used in the experiment are given as the

columns of this array. There is no combination in which two factors appear both at level 2 or 3. But other than that, with respect to any two factors all combinations of levels that we did not exclude appear exactly once in these columns. This is an example of what is known as an incomplete orthogonal array.

We would like to point out that in the above example the design can be completed to an orthogonal array by adding the following four (undesirable) combinations:

2	2	3	3
2	3	2	3
2	3	3	2

Although the name 'incomplete orthogonal array' may give rise to the idea that it can always be completed to an orthogonal array (without increasing the frequency of level combinations that were not excluded), this is not the case. There are two reasons why it may not be possible to do this. It is possible that an orthogonal array with the required parameters does not exist, but it may also be that the structure of the incomplete orthogonal array does not permit such a completion even though an orthogonal array with the desired parameters does exist. For more details we refer to Hedayat and Stufken (1986a).

In Section 2 we will formally introduce incomplete orthogonal arrays. In Section 3 we will study some of the statistical aspects of these arrays. Section 4 will give a brief discussion of possible future research projects in this subject. For the mathematical properties of incomplete orthogonal arrays we refer the reader to Horton (1974), Maurin (1985) and Hedayat and Stufken (1986a).



## 2. DEFINITIONS AND NOTATIONS

In this section we will give a formal definition of an incomplete orthogonal array. Throughout the paper  $S$  will denote a set of  $s$  symbols, while  $H$  will be a subset of  $S$  of cardinality  $h$ .

Definition 2.1: A  $k \times N$  array based on  $S$  is called an incomplete orthogonal array based on  $S$  and  $H$  and of strength  $t$  if the columns of any  $t \times N$  subarray contain each element from  $S^t - H^t$  equally often, say  $\lambda$  times, while those from  $H^t$  do not appear at all.

An immediate consequence of the definition is that the equality  $N = \lambda(s^t - h^t)$  must hold. The integer  $\lambda$  is also called the index of the array. We will denote these arrays by  $IOA(N, k, (s, h), t)$ .

For comparison we recall the definition of an orthogonal array.

Definition 2.2: A  $k \times N$  array based on  $S$  is called an orthogonal array of strength  $t$  if the columns of any  $t \times N$  subarray contain each element of  $S^t$  equally often, say  $\lambda$  times.

Here we see immediately that  $N = \lambda s^t$ . We denote these arrays by  $OA(N, k, s, t)$ .

As in the study of orthogonal arrays, an interesting question is to find the largest value of  $k$  for which an  $IOA(N, k, (s, h), t)$  exists, for given  $N, s, h$  and  $t$ . Hedayat and Stufken (1986a) showed that the inequality

$$k \leq s/h + t - 1$$

must always be satisfied.

We would like to point out that incomplete orthogonal arrays are a heavily structured type of balanced arrays. The latter arrays were introduced by Chakravarti (1956) and many interesting results on it were obtained by Srivastava (1972) and Rafter and Seiden (1974). Srivastava and her co-workers, as well as several Japanese statisticians, have made valuable contributions to the area of balanced arrays.

It is easy to show that an orthogonal array of strength  $t$  is also of strength  $t' \leq t$ . However for incomplete orthogonal arrays such a property does no longer hold. In any  $t'$  rows of an  $\text{IOA}(\lambda(s^t - h^t), k, (s, h), t)$ ,  $t \geq t'$ , it can easily be shown that any  $t'$ -tuple from  $H^{t'}$  appears  $\lambda(s^{t-t'} - h^{t-t'})$  times, while those from  $S^{t'} - H^{t'}$  appear  $\lambda s^{t-t'}$  times each. The array is thus still a heavily structured balanced array of strength  $t'$ , but with our definition of an incomplete orthogonal array there is for any array at most one strength for which the requirements in the definition are satisfied.

### 3. STATISTICAL ASPECTS OF INCOMPLETE ORTHOGONAL ARRAYS

In this section we will prove a general result on the use of incomplete orthogonal arrays of strength 2 as fractional factorial designs under the orthogonal polynomial model. We will illustrate this result with two detailed examples. We will assume that the reader is familiar with the standard terminology for factorial designs. If not, a useful reference is Raktoe, Hedayat and Federer (1981).

Suppose that under the orthogonal polynomial model in a symmetrical factorial design we can neglect all the interaction effects. It is well known that if a fractional factorial design in the form of an orthogonal array of strength 2 is used, the general mean and all main effects are (orthogonally) estimable. The following theorem gives a comparable result for incomplete orthogonal arrays.

Theorem 3.1: If we use an  $\text{IOA}(\lambda(s^2 - h^2), k, (s, h), 2)$  as a fractional factorial design under the orthogonal polynomial model and under the assumption that the only non-zero factorial effects are the general mean and the main effects, then

- (1) if  $k < s/h + 1$  all effects are estimable;

- (ii) if  $k = s/h + 1$  we add one more run in which all factors are at one of the  $h$  levels of  $H$  and reach then the same conclusion as in (i).

(Recall from Section 2 that  $k \leq s/h + 1$ .)

Proof: Let  $A_1$  be the incomplete orthogonal array based on  $S$  and  $H$  as in the statement of the Theorem. Let  $A_2$  be the fractional factorial design with all  $h^k$  runs from  $H^k$ . Let  $A$  be the juxtaposition of  $A_1$  and  $A_2$ . If we now let  $X, X_1$  and  $X_2$  be the design matrices corresponding to  $A, A_1$  and  $A_2$  respectively, under the model as described in the Theorem, then we can write

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Our first claim is that  $X$  is of full column rank, i.e.,  $r(X) = k(s-1) + 1$ . An easy way to see this is as follows: look at  $A^*$ , the juxtaposition of  $h^{k-2}$  replications of  $A_1$  and  $\lambda$  replications of  $A_2$ . Then  $A^*$  is an orthogonal array of strength 2, implying that its design matrix  $X^*$  under the above model is of rank  $k(s-1) + 1$ . But the rows in  $X^*$  are just replications of the rows in  $X$ . Hence  $r(X) = r(X^*)$  which shows the claim. We would like to show that  $r(X_1) = k(s-1) + 1$ . For this now it suffices to show that the row space of  $X_2$  is a subspace of the row space of  $X_1$ . Choose an arbitrary row in  $X_2$ , say  $r$ , which corresponds to a treatment combination, say  $(\alpha_1, \dots, \alpha_k)$ , in which all levels are from  $H$ . In  $A_1$  there are  $\lambda(s-h)$  runs in which the  $i^{\text{th}}$  factor is at level  $\alpha_i$ . With  $u = \lambda(s-h)$  we denote the corresponding rows in  $X_1$  by  $r_1^{(i)}, \dots, r_u^{(i)}$ ,  $1 \leq i \leq k$ . The number of runs in  $A_1$  in which no factor is at a level from  $H$  equals  $\lambda(s^2 - h^2) - \lambda kh(s-h) = \lambda(s-h)(s+h-kh)$ . If we denote this number by  $v$  then  $v > 0$  if and only if  $k < s/h + 1$ . Assuming that this holds let  $r_1, \dots, r_v$  be the rows in  $X_1$  corresponding to these runs. We claim now that

$$(3.1) \quad \sum_{i=1}^k \sum_{j=1}^u r_j^{(i)} = ur + \frac{(k-1)u}{v} \sum_{i=1}^v r_i.$$

Notice that this claim would establish (i), since the arbitrarily selected row  $r$  from  $X_2$  is expressed as a linear combination of rows in  $X_1$ .

Let  $M = (m_{ij})$ ,  $0 \leq i, j \leq s-1$  be the orthogonalized matrix corresponding to a one-factor orthogonal polynomial model. The entry in  $X$  corresponding to a run  $(\beta_1, \dots, \beta_k)$  and an effect  $\phi_1^0 \dots \phi_{\ell-1}^0 \phi_{\ell}^j \phi_{\ell+1}^0 \dots \phi_k^0$ ,  $1 \leq \ell \leq k$ ,  $0 \leq j \leq s-1$  is given by  $m_{\beta_{\ell}j}$ . (This is under the assumption that  $M$  is not normalized and that its first column is  $(1, \dots, 1)'$ . If  $M$  is normalized we would have to add everywhere a constant term, but other than that the proof can be continued unchanged. See also Hedayat and Stufken (1986a)).

In the runs corresponding to the rows in the left hand side of (3.1) factor  $\ell$  is  $u$  times at level  $\alpha_{\ell}$  and  $\lambda(k-1)$  times at level  $\gamma$  for any  $\gamma \in S - H$ . Hence the coordinate of the left hand side in (3.1) corresponding to the effect

$\phi_1^0 \dots \phi_{\ell-1}^0 \phi_{\ell}^j \phi_{\ell+1}^0 \dots \phi_k^0$  equals

$$(3.2) \quad um_{\alpha_{\ell}j} + \lambda(k-1) \sum_{\gamma \in S-H} m_{\gamma j}.$$

In the runs corresponding to  $r_1, \dots, r_v$  factor  $\ell$  is  $\lambda(s+h-hk)$  times at level  $\gamma \in S - H$ . In the run corresponding to  $r$  factor  $\ell$  is at level  $\alpha_{\ell}$ . Hence the coordinate of the right hand side in (3.1) corresponding to the above effect equals

$$(3.3) \quad um_{\alpha_{\ell}j} + \frac{(k-1)u}{v} \lambda(s+h-hk) \sum_{\gamma \in S-H} m_{\gamma j}.$$

It is clear that (3.2) and (3.3) are the same, which establishes (3.1) and thus (i).

For (ii) the problem is that  $v = 0$ , and every run in  $A_1$  contains exactly one element from  $H$ . However choosing an arbitrary row  $r$  from  $X_2$  and defining the rows  $r_j^{(i)}$  as above we obtain that the coordinate of

$$\sum_{i=1}^k \sum_{j=1}^u r_j^{(i)} - ur$$

corresponding to the same factorial effect as above equals

$$\lambda(k-1) \sum_{\gamma \in S-H} m_{\gamma j}.$$

Since this does not depend on  $r$  we see that if we add one row from  $X_2$  to  $X_1$  all other rows from  $X_2$  can be obtained as a linear combination of the selected row in  $X_2$  and the rows in  $X_1$ . That shows (ii) and concludes the proof of Theorem 3.1  $\square$

We will now give two examples, the first of which relates to the situation in Theorem 3.1 (ii), the second to part (i) of that theorem.

Example 3.1: Let  $s = 4$ ,  $h = 2$ ,  $k = 3$ . Indeed  $k = s/h + 1$ . Consider the incomplete orthogonal array from Section 1.:

```

0 0 0 0 1 1 1 1 2 2 3 3
0 1 2 3 0 1 2 3 0 1 0 1
2 3 1 0 3 2 0 1 1 0 0 1

```

The design matrix  $X_1$  for this fractional factorial design under the model in Theorem 3.1 is given below. For its computation we used the matrix  $M$  given by

$$M = \begin{bmatrix} 1 & -3 & 1 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

We obtained:

$$X_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 & -1 & -3 & 1 & -1 & 1 & -1 & -3 \\ 1 & -3 & 1 & -1 & -1 & -1 & 3 & 3 & 1 & 1 \\ 1 & -3 & 1 & -1 & 1 & -1 & -3 & -1 & -1 & 3 \\ 1 & -3 & 1 & -1 & 3 & 1 & 1 & -3 & 1 & -1 \\ 1 & -1 & -1 & 3 & -3 & 1 & -1 & 3 & 1 & 1 \\ 1 & -1 & -1 & 3 & -1 & -1 & 3 & 1 & -1 & -3 \\ 1 & -1 & -1 & 3 & 1 & -1 & -3 & -3 & 1 & -1 \\ 1 & -1 & -1 & 3 & 3 & 1 & 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & -3 & -3 & 1 & -1 & -1 & -1 & 3 \\ 1 & 1 & -1 & -3 & -1 & -1 & 3 & -3 & 1 & -1 \\ 1 & 3 & 1 & 1 & -3 & 1 & -1 & -3 & 1 & -1 \\ 1 & 3 & 1 & 1 & -1 & -1 & 3 & -1 & -1 & 3 \end{bmatrix}$$

The columns of  $X_1$  correspond, in the given order, to the following factorial effects:

$$\phi_1^0 \phi_2^0 \phi_3^0, \phi_1^1 \phi_2^0 \phi_3^0, \phi_1^2 \phi_2^0 \phi_3^0, \phi_1^3 \phi_2^0 \phi_3^0, \phi_1^0 \phi_2^1 \phi_3^0, \phi_1^0 \phi_2^2 \phi_3^0, \phi_1^0 \phi_2^3 \phi_3^0, \phi_1^0 \phi_2^0 \phi_3^1, \\ \phi_1^0 \phi_2^0 \phi_3^2, \phi_1^0 \phi_2^0 \phi_3^3.$$

That this matrix is not of full column rank becomes clear if we look at the information matrix  $X_1'X_1$ . This matrix is displayed below.

$$X_1'X_1 = \begin{bmatrix} 12 & -8 & 0 & 4 & -8 & 0 & 4 & -8 & 0 & 4 \\ -8 & 60 & -4 & 0 & -16 & 0 & 8 & -16 & 0 & 8 \\ 0 & -4 & 12 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -8 & 60 & 8 & 0 & -4 & 8 & 0 & -4 \\ -8 & -16 & 0 & 8 & 60 & -4 & 0 & -16 & 0 & 8 \\ 0 & 0 & 0 & 0 & -4 & 12 & -8 & 0 & 0 & 0 \\ 4 & 8 & 0 & -4 & 0 & -8 & 60 & 8 & 0 & -4 \\ -8 & -16 & 0 & 8 & -16 & 0 & 8 & 60 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 12 & -8 \\ 4 & 8 & 0 & -4 & 8 & 0 & -4 & 0 & -8 & 60 \end{bmatrix}$$

It is easy to verify that the vector  $(5, 2, 0, -1, 2, 0, -1, 2, 0, -1)'$  is in the null space of this matrix.

If we add one more run to our fractional factorial design, a run in which each factor appears at a level from  $H = \{2, 3\}$  the new design matrix, say  $X$ , is of full column rank. As an example, suppose we add the run  $(2, 2, 2)$ . This corresponds to adding the row vector

(1,1,-1,-3,1,-1,-3,1,-1,-3)

to the previous design matrix. Computing  $X'X$  gives

$$X'X = \begin{bmatrix} 13 & -7 & -1 & 1 & -7 & -1 & 1 & -7 & -1 & 1 \\ -7 & 61 & -5 & -3 & -15 & -1 & 5 & -15 & -1 & 5 \\ -1 & -5 & 13 & -5 & -1 & 1 & 3 & -1 & 1 & 3 \\ 1 & -3 & -5 & 69 & 5 & 3 & 5 & 5 & 3 & 5 \\ -7 & -15 & -1 & 5 & 61 & -5 & -3 & -15 & -1 & 5 \\ -1 & -1 & 1 & 3 & -5 & 13 & -5 & -1 & 1 & 3 \\ 1 & 5 & 3 & 5 & -3 & -5 & 69 & 5 & 3 & 5 \\ -7 & -15 & -1 & 5 & -15 & -1 & 5 & 61 & -5 & -3 \\ -1 & -1 & 1 & 3 & -1 & 1 & 3 & -5 & 13 & -5 \\ 1 & 5 & 3 & 5 & 5 & 3 & 5 & -3 & -5 & 69 \end{bmatrix}$$

This matrix is of full rank, and its inverse is given by:

$$(X'X)^{-1} = \frac{1}{800} \begin{bmatrix} 125 & 35 & 25 & -5 & 35 & 25 & -5 & 35 & 25 & -5 \\ 35 & 27 & 15 & -1 & 16 & 10 & -3 & 16 & 10 & -3 \\ 25 & 15 & 75 & 5 & 10 & 0 & -5 & 10 & 0 & -5 \\ -5 & -1 & 5 & 13 & -3 & -5 & -1 & -3 & -5 & -1 \\ 35 & 16 & 10 & -3 & 27 & 15 & -1 & 16 & 10 & -3 \\ 25 & 10 & 0 & -5 & 15 & 75 & 5 & 10 & 0 & -5 \\ -5 & -3 & -5 & -1 & -1 & 5 & 13 & -3 & -5 & -1 \\ 35 & 16 & 10 & -3 & 16 & 10 & -3 & 27 & 15 & -1 \\ 25 & 10 & 0 & -5 & 10 & 0 & -5 & 15 & 75 & 5 \\ -5 & -3 & -5 & -1 & -3 & -5 & -1 & -1 & 5 & 13 \end{bmatrix}$$

Example 3.2: Let  $s = 4$ ,  $h = 2$  and  $k = 2$ . Thus  $k < s/h + 1$ .

Consider the following incomplete orthogonal array:

```
0 0 0 0 1 1 1 1 2 2 3 3
0 1 2 3 0 1 2 3 0 1 0 1
```

We will in this example, since we will not only be interested in the rank of the design matrix but will give some more detailed computations, use the normalized form of  $M$ , i.e.,

$$M = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} & \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} & -\frac{1}{2} & \frac{3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} & -\frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} & \frac{1}{2} & \frac{1}{2\sqrt{5}} \end{bmatrix}$$

The design matrix  $X_1$  under the model in Theorem 3.1 and the information matrix  $X_1'X_1$  as well as its inverse  $(X_1'X_1)^{-1}$  are displayed below.

$$4\sqrt{5} X_1 = \begin{bmatrix} 0 & 0 & \sqrt{5} & -3 & \sqrt{5} & -1 & -3 & \sqrt{5} & -1 \\ 0 & 1 & \sqrt{5} & -3 & \sqrt{5} & -1 & -1 & -\sqrt{5} & 3 \\ 0 & 2 & \sqrt{5} & -3 & \sqrt{5} & -1 & 1 & -\sqrt{5} & -3 \\ 0 & 3 & \sqrt{5} & -3 & \sqrt{5} & -1 & 3 & \sqrt{5} & 1 \\ 1 & 0 & \sqrt{5} & -1 & -\sqrt{5} & 3 & -3 & \sqrt{5} & -1 \\ 1 & 1 & \sqrt{5} & -1 & -\sqrt{5} & 3 & -1 & -\sqrt{5} & 3 \\ 1 & 2 & \sqrt{5} & -1 & -\sqrt{5} & 3 & 1 & -\sqrt{5} & -3 \\ 1 & 3 & \sqrt{5} & -1 & -\sqrt{5} & 3 & 3 & \sqrt{5} & 1 \\ 2 & 0 & \sqrt{5} & 1 & -\sqrt{5} & -3 & -3 & \sqrt{5} & -1 \\ 2 & 1 & \sqrt{5} & 1 & -\sqrt{5} & -3 & -1 & -\sqrt{5} & 3 \\ 3 & 0 & \sqrt{5} & 3 & \sqrt{5} & 1 & -3 & \sqrt{5} & -1 \\ 3 & 1 & \sqrt{5} & 3 & \sqrt{5} & 1 & -1 & -\sqrt{5} & 3 \end{bmatrix}$$

The columns correspond to  $\phi_1^0\phi_2^0, \phi_1^1\phi_2^0, \phi_1^2\phi_2^0, \phi_1^3\phi_2^0, \phi_1^0\phi_2^1, \phi_1^1\phi_2^1, \phi_1^2\phi_2^1, \phi_1^3\phi_2^1$  in that order.



$$X_1'X_1 = \begin{bmatrix} \frac{3}{4} & \frac{-1}{2\sqrt{5}} & 0 & \frac{1}{4\sqrt{5}} & \frac{-1}{2\sqrt{5}} & 0 & \frac{1}{4\sqrt{5}} \\ \frac{-1}{2\sqrt{5}} & \frac{3}{4} & \frac{-1}{4\sqrt{5}} & 0 & \frac{-1}{5} & 0 & \frac{1}{10} \\ 0 & \frac{-1}{4\sqrt{5}} & \frac{3}{4} & \frac{-1}{2\sqrt{5}} & 0 & 0 & 0 \\ \frac{1}{4\sqrt{5}} & 0 & \frac{-1}{2\sqrt{5}} & \frac{3}{4} & \frac{1}{10} & 0 & \frac{-1}{20} \\ \frac{-1}{2\sqrt{5}} & \frac{-1}{5} & 0 & \frac{1}{10} & \frac{3}{4} & \frac{-1}{4\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{4\sqrt{5}} & \frac{3}{4} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{4\sqrt{5}} & \frac{1}{10} & 0 & \frac{-1}{20} & 0 & \frac{-1}{2\sqrt{5}} & \frac{3}{4} \end{bmatrix}$$

$$(X_1'X_1)^{-1} = \begin{bmatrix} 2 & \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{19}{10} & \frac{1}{2\sqrt{5}} & \frac{-1}{5} & \frac{4}{5} & 0 & \frac{-2}{5} \\ 0 & \frac{1}{2\sqrt{5}} & \frac{3}{2} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{-1}{\sqrt{5}} & \frac{-1}{5} & \frac{1}{\sqrt{5}} & \frac{8}{5} & \frac{-2}{5} & 0 & \frac{1}{5} \\ \frac{2}{\sqrt{5}} & \frac{4}{5} & 0 & \frac{-2}{5} & \frac{19}{10} & \frac{1}{2\sqrt{5}} & \frac{-1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{5}} & \frac{3}{2} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{-2}{5} & 0 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{\sqrt{5}} & \frac{8}{5} \end{bmatrix}$$

It is now a simple exercise to compute the BLUE of each of the factorial effects. These are given below, with  $Y_{\alpha\beta}$  referring to the observation under run  $(\alpha, \beta)$ .

$$\hat{\phi}_{1\phi_2}^{0\ 0} = \frac{1}{2}(Y_{02} + Y_{03} + Y_{12} + Y_{13} + Y_{20} + Y_{21} + Y_{30} + Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{1\ 0} = \frac{1}{4\sqrt{5}} (-5Y_{00} - 5Y_{01} - Y_{02} - Y_{03} - 3Y_{10} - 3Y_{11} + Y_{12} + Y_{13} + 2Y_{20} + 2Y_{21} + 6Y_{30} + 6Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{2\ 0} = \frac{1}{4}(Y_{00} + Y_{01} + Y_{02} + Y_{03} - Y_{10} - Y_{11} - Y_{12} - Y_{13} - 2Y_{20} - 2Y_{21} + 2Y_{30} + 2Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{3\ 0} = \frac{1}{2\sqrt{5}} (-Y_{02} - Y_{03} + 2Y_{10} + 2Y_{11} + Y_{12} + Y_{13} - 3Y_{20} - 3Y_{21} + Y_{30} + Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{0\ 1} = \frac{1}{4\sqrt{5}} (-5Y_{00} - 3Y_{01} + 2Y_{02} + 6Y_{03} - 5Y_{10} - 3Y_{11} + 2Y_{12} + 6Y_{13} - Y_{20} + Y_{21} - Y_{30} + Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{0\ 2} = \frac{1}{4} (Y_{00} - Y_{01} - 2Y_{02} + 2Y_{03} + Y_{10} - Y_{11} - 2Y_{12} + 2Y_{13} + Y_{20} - Y_{21} + Y_{30} - Y_{31})$$

$$\hat{\phi}_{1\phi_2}^{0\ 3} = \frac{1}{2\sqrt{5}} (2Y_{01} - 3Y_{02} + Y_{03} + 2Y_{11} - 3Y_{12} + Y_{13} - Y_{20} + Y_{21} - Y_{30} + Y_{31})$$

#### 4. CONCLUDING REMARKS

In the previous sections we indicated situations for which the use of incomplete orthogonal arrays as fractional factorial designs might be considered and gave a useful statistical property for such arrays of strength 2. However, we are well aware that many interesting questions remain unsolved. A study of the statistical properties of incomplete orthogonal arrays of arbitrary strength  $t$  would certainly be welcome. The available results in the literature on the construction of these arrays are also rather meager. It would also be nice if we could replace the added treatment combination in Theorem 3.1 (ii) by one in which at most one factor appears at a level from  $H$ . We do not know whether this can be achieved or not.

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